APPENDIX

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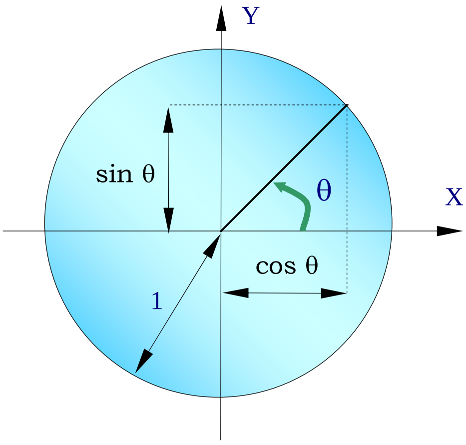
Plane Trigonometry

In a circle with radius 1 the cos and the sin of an angle θ are respectively the length of the projections on the X and Y axis of any radius making an angle θ with the X axis.



THE BASIC FORMULA OF TRIGONOMETRY

The basic formula is valid for any value of θ:

$$\cos^2 \theta + \sin^2 \theta = 1$$

values derived from cos and sin :

$$\sec \theta = \frac{1}{\cos \theta} \qquad \text{cosec } \theta = \frac{1}{\sin \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \text{cotg } \theta = \frac{\cos \theta}{\sin \theta}$$

$$\tan \theta = \frac{1}{\text{cotg } \theta}$$

Units for angles

Degrees: from 0° to 360°, is the most currently used unit for practical purposes
Radians: from 0 to 2π, used in mathematics, this unit is equal to the arc length of the portion of the unit circle cut off by the angle θ.

Conversion between units: $1° = \frac{180}{\pi}$ rad

Addition formulas

The addition formulas are valid for any x and y

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

Reverse formulas

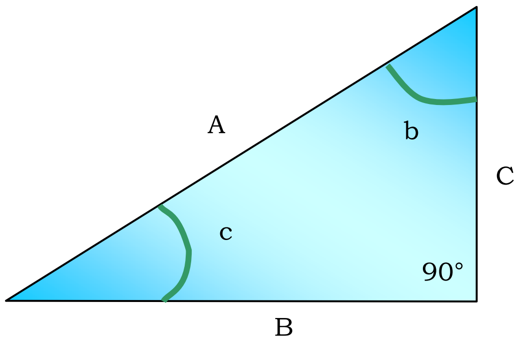
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = -2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

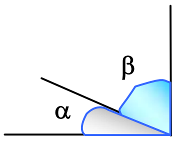
Formulas for plane right-angled triangles



$A^2=B^2+C^2$	$c+b=90°$
$B=A\cos c$	$B=A\sin b$
$C=A\sin c$	$C=A\cos b$
$\operatorname{tg} c=\frac{C}{B}$	$\operatorname{tg} b=\frac{B}{C}$

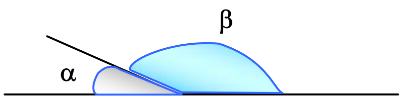
PLANE GEOMETRY

Complementary angles



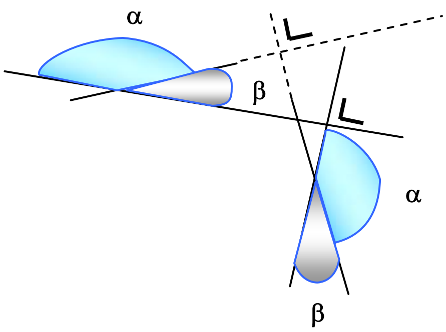
The angles α and β are complementary because α+β=90° , α is called the complement of β, and vice versa.

Supplementary angles



The angles α and β are supplementary because α+β=180°, ° , α is called the supplement of β, and vice versa.

Apparented angles

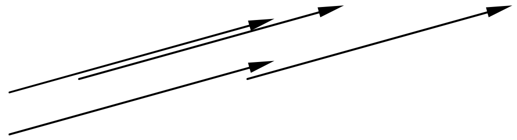


Angles of which the legs are perpendicular to each other are equal or supplementary to each other.

VECTORS

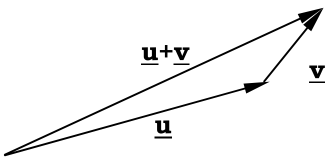
Definition of a vector

A vector is completely defined by tree properties: length, orientation, direction



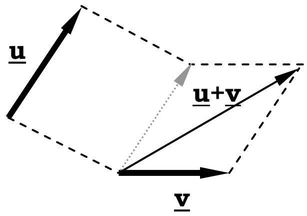
The arrows above so represent the same vector, because each arrow has the same orientation, length and direction. An individual arrow is called a representative of the vector. The length of vector **v** is represented as |**v**| and is always a positive number.

Sum of two vectors



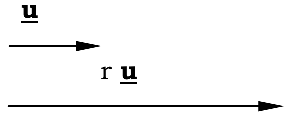
When adding two vectors we "close" the triangle formed by the vectors, here we have applied the parallelogram rule in order to find their most appropriate representatives

The parallelogram rule



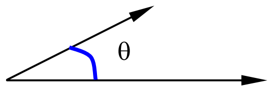
Above we used the parallelogram rule to find the most appropriate representative of a vector in order to perform an addition.

Multiplication of a vector with a number



When we multiply a vector with a number r we only multiply the length of the vector with r, the direction of the vector remains the same if r is positive and is reversed if negative, the orientation remains the same. The result is always a vector.

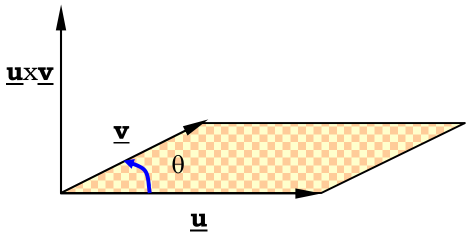
The Dot product of two vectors



v.u= |v||u| cos θ, the result is always a number. If θ=90° then **u.v**=0

Space Geometry

Cross product of two vectors



The result is a vector perpendicular on both **u** and **v**, with length **|vxu|= |v||u| sin** θ, which is equal to the surface of the parallelogram formed by **u** and **v**. The sense of **u**x**v** is obtained by the right hand rule.

If θ =0° then **v**x**u**=0. This is the case with parallel vectors. In consequence of this **u.**(**u**x**v**)=0

Some vector algebra

r(**u**+**v**)=r **u**+ r **v**

v+**u**=**u**+**v**

w.(**u**+**v**)= **w.u**+**w.v**

v.u=u.v

vx**u**=-**u**x**v**

Mathematical derivation of Astronavigal Formulas

First formula

Consider three vectors in space **A, B, C**, we can prove that:

Ax(**B**x**C**)=**B**.(**A**.**C**)-**C**.(**A**.**B**) (1)

From (1) we derive:

(**A**x**B**)x(**A**x**C**)=**A**.(**A**x**B**.**C**)-**C**(**A**x**B**.**B**)=**A**.(**A**x**B**.**C**)+**0**=**A**.(**A**x**B**.**C**)

(**A**x**B**)x(**A**x**C**)= **A**.(**A**x**B**.**C**) (1)

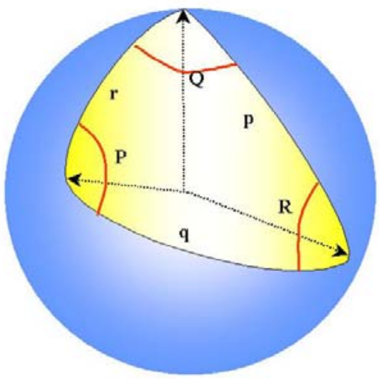
the result establishes that this vector is always parallel to **A**.

Second formula:

(**A**x**B**)x(**A**x**C**)=(**B**.**C**)(**A**.**A**)-(**A**.**C**).(**A**.**B**) (2)

Application to spherical triangles

Consider three points on a sphere with unit radius, these points are the endings of the unit vectors **A, B, C**



These endings form a spherical triangle, on which we apply formulas (1)&(2) On the vectors **A, B, C** we are able to prove some properties of spherical triangles. The angles in space between the vectors are p, q, r which are also the arc lengths of the triangle. The angles between the sides of the triangle are P, Q, R named the same as the opposite arc.

With formula (1):

(**A**x**B**)x(**A**x**C**)= **A**.(**A**x**B**.**C**)   **//A**

For the vectors from our spherical triangle from fig 1:

(sin q) **1ab** x (sin r) **1ac** = A(**A**x**B**.**C**).**1aa**

(sin q).(sin r).(cos P).**1aa**= A(**A**x**B**.**C**).**1aa**

by cyclic permutation we obtain:

$$\frac{\sin R}{\sin r} = \frac{\sin P}{\sin p} = \frac{\sin Q}{\sin q} \quad (3)$$

With formula (2)

For the vectors from our spherical triangle from fig 1:

(**A**x**B**)x(**A**x**C**)=(**B**.**C**)**A**.**A**-(**A**.**C**).(**A**.**B**)

(sin q) . (sin r) . cos (P) = cos p - cos r cos q

cos p = cos r cos q + sin r sin q cos P (4)

With formula (3) and (4) we are now able to compute the arcs and angles of every spherical triangle of a unit sphere. Formulas (3) and (4) do not have the appropriate form for astronavigal computations. To make computations easier formula (4) is put in the haversine form. The haversine is a tabulated function and has the following definition:

Haversine of x: $\text{hav } x = \frac{1-\cos x}{2}$

Derivation of the haversine formula:

From formula (4) we derive

1-cos p = 1- cos r cos q - sin q sin r . cos P
1-cos p = 1- cos r cos q - sin q sin r . cos P-sin r sin q + sin r sin q
1-cos p = 1- sin r sin q - cos r cos q. + sin r sin q - sin r . sin q . cos P
1-cos p = 1- cos(q-r) + sin r sin q - sin r sin q . cos P
1-cos p = 1- cos(q-r) + sin r sin q (1-cos P)

$$\frac{1-\cos p}{2} = \frac{1-\cos(q-r)}{2} + \sin r \sin q \frac{(1-\cos P)}{2}$$

hav p = hav (q-r) + sin r sin q hav P

Third Formula

$\cos r = \cos p \cos q + \sin p \sin q \cos R$ (1)

$\cos p = \cos r \cos q + \sin r \sin q \cos P$ (2) and $\sin p = \frac{\sin r \sin P}{\sin R}$ (3)

substitution of (2) and (3) in (1) gives:

$\cos r = (\cos r \cos q + \sin r \sin q \cos P) \cos q + \frac{\sin r \sin P}{\sin R} \sin q \cos R$

$\cos r = \cos r \cos^2 q + \sin r \sin q \cos P \cos q + \sin r \sin P \sin q \text{ cotg } R$

$\cos r - \cos r \cos^2 q = \sin q \sin r \cos P \cos q + \sin q \sin r \sin P \text{ cotg } R$

$(1 - \cos^2 q) \cos r = \sin q \sin r (\cos P \cos q + \sin P \text{ cotg } R)$

$\sin^2 q \cos r = \sin q \sin r (\cos P \cos q + \sin P \text{ cotg } R)$

$\frac{\sin^2 q \cos r}{\sin q \sin r} = \frac{\sin q \sin r}{\sin q \sin r} (\cos P \cos q + \sin P \text{ cotg } R)$

$\sin q \text{ cotg } r = \cos P \cos q + \sin P \text{ cotg } R$

or :

$$\textbf{cotg } R = \frac{(\sin q \text{ cotg } r - \cos P \cos q)}{\sin P}$$

Used shape for tabulation:

Dividing both sides by sin q gives

$\frac{\text{cotg } R}{\sin q} = \frac{\text{cotg } r}{\sin P} - \frac{\cos P \cos q}{\sin P \sin q}$

or :

$$\textbf{cotg } R \textbf{ cosec } q = \textbf{cotg } r \textbf{ cosec } P - \textbf{cotg } P \textbf{ cotg } q$$

Formulas for right-angled spherical triangles

A right-angled triangle is a spherical triangle that has an angle equal to 90°.

Thus the formulas are obtained by setting R=90°, hence sin R=1 and cos R= 0.

The sine formula

The sine formula becomes:

$$\frac{1}{\sin r} = \frac{\sin P}{\sin p} = \frac{\sin Q}{\sin q}$$

solving regarding to sin q and sin p gives:

sin q = sin Q sin r

And :

sin p = sin P sin r

The cosine formula

With the cosine formula applied on sides p and q we find:

cos p = cos q cos r + sin q sin r cos P (1)

cos q = cos p cos r + sin p sin r cos Q (2)

cos r = cos p cos q hence $\cos p = \frac{\cos r}{\cos q}$ (3) as cos R = 0

we substitute (3) in (1)

$\frac{\cos r}{\cos q} = \cos q \cos r + \sin r \sin q \cos P$ which we solve with respect to cos P

$$\cos P = \frac{\cos r - \cos r \cos^2 q}{\sin r \sin q \cos q} = \frac{\cos r(1-\cos^2 q)}{\sin r \sin q \cos q} = \frac{\cos r \sin^2 q}{\sin r \sin q \cos q} = \frac{\cos r \sin q}{\sin r \cos q}$$

which gives :

cos P = cotg r tg q

And similarly by substitution of (3) in (2) we obtain:

cos Q = cotg r tg p

The tangent formula

Dividing the cosine formulas with their respective sine formula gives:

$\frac{\cos P}{\sin P} = \text{cotg } r \ \text{tg } q \ \frac{\sin r}{\sin p}$ gives $\text{cotg } P = \cos r \ \frac{\text{tg } q}{\sin p}$

$\frac{\cos Q}{\sin Q} = \text{cotg } r \ \text{tg } p \ \frac{\sin r}{\sin q}$ gives $\text{cotg } Q = \cos r \ \frac{\text{tg } p}{\sin q}$

multiplying side by side gives:

$\text{cotg } P \ \text{cotg } Q = \frac{\cos^2 r \ \text{tg } p \ \text{tg } q}{\sin p \sin q}$ as $\text{tg } p = \frac{\sin p}{\cos p}$ and $\text{tg } q = \frac{\sin q}{\cos q}$

hence :

$\text{cotg } P \ \text{cotg } Q = \frac{\cos^2 r}{\cos p \cos q}$ as $\cos r = \cos p \cos q$

$\text{cotg } P \ \text{cotg } Q = \cos r$

or:

$$\cos r = \text{cotg } P \ \text{cotg } Q$$

Ex Meridian formula

From the general formula $\sin h = \sin l \sin \delta + \cos l \cos \delta \cos LHA$

it follows at meridional passage when LHA is 0° that :

$\sin h = \sin l \sin \delta + \cos l \cos \delta \cos LHA$ or $\sin h = \cos (l - \delta)$ and $\cos h = \sin (l - \delta)$

In which Δt is a time just before or just after the meridional passage which corresponds respectively to LHA = 0° - ΔA or LHA = 0°+ΔA.

ΔA is Δt converted to arc.

Then:

$\sin (h+\Delta h) = \sin l \sin \delta + \cos l \cos \delta \cos \Delta A$

or

$\sin h \cos \Delta h + \cos h \sin \Delta h = \sin l \sin \delta + \cos l \cos \delta \cos \Delta A$

or

$\cos (l - \delta) \cos \Delta h + \sin (l - \delta) \sin \Delta h = \sin l \sin \delta + \cos l \cos \delta \cos \Delta A$

or

$$\sin \Delta h = \frac{\sin l \sin \delta - \cos (l - \delta) \cos \Delta h + \cos l \cos \delta \cos \Delta A}{\sin (l - \delta)}$$

when Δh is small enough then $\cos \Delta h = 1$ and $\sin \Delta h = \Delta h$ (1)

then

$$\Delta h = \frac{\sin l \sin \delta - \cos (l - \delta) + \cos l \cos \delta \cos \Delta A}{\sin (l - \delta)}$$

and

$$\Delta h = \frac{\sin l \sin \delta - (\cos l \cos \delta + \sin l \sin \delta) + \cos l \cos \delta \cos \Delta A}{\sin (l - \delta)}$$

and

$$\Delta h = \frac{\cos l \cos \delta (1 - \cos \Delta A)}{\sin (l - \delta)}$$ when A = 0,25° and if Δh is expressed in arc seconds

$$\Delta h = 1{,}96345 \frac{\cos l \cos \delta}{\sin (l - \delta)}$$

(1) when h is expressed in radians

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